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DYNAMICAL MODELLING OF LINEAR DISCRETE-CONTINUOUS SYSTEMS†

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Problems that arise when constructing and analysing dynamical models of linear discrete-continuous systems, which are represented as matrices of quasi-rational fractions, that is to say, quotients of quasi-polynomials, are investigated. Theorems on the roots of the characteristic quasi-polynomial and the stability of the dynamical models are formulated. As an example, the dynamical modelling of an elastic link of a manipulator is considered. © 2000 Elsevier Science Ltd. All rights reserved.

1. EQUATIONS OF MOTION

In control theory and stability analysis for systems with deformable controlled objects [1–4], mechanical models of such systems are represented by a combination of absolutely rigid and elastic bodies dynamically linked through interfaces. The equations of motion of such systems (discrete-continuous systems (DCS) in the terminology of [1]) comprise the system of ordinary differential equations of motion of the absolutely rigid bodies, the partial differential equations of the motion of the elastic bodies with boundary conditions, linkage conditions in the form of point boundary forces and couples on the absolutely rigid bodies, as well as initial conditions.

We consider the equations of motion of a DCS in general form, expressed in dimensionless form

$$F[x(t), y(t), \dot{y}(t), \ddot{y}(t), n(t); \varepsilon] = 0, \quad \varepsilon \ll 1 \tag{1.1}$$

$$Gw(z,t) = V(z, x, y, \dot{y}, \ddot{y}; \varepsilon) \tag{1.2}$$

$$z = 0:\; w(0,t) = P_1,\;\; w'(0,t)E = P_2;\;\; z = 1:\; w(1,t) = P_3,\;\; w'(1,t)E = P_4 \tag{1.3}$$

$$\begin{gathered} E = [1, \dots 1]^T, ()' = \partial/\partial z,\;\; P_j = P_j(y; \varepsilon),\;\; j = 1, \dots 4 \\ n(t) = gf(w(z,t)) \end{gathered} \tag{1.4}$$

$$t = 0:\; y(0) = y_0,\;\; \dot{y}(0) = \dot{y}_0,\;\; w(z,0) = w_0(z),\;\; \partial w(z,0)/\partial t = \dot{w}_0(z) \tag{1.5}$$

where (1.1) are the ordinary differential equations of motion of the absolutely rigid bodies, (1.2) are the partial differential equations of motion of the elastic bodies, (1.3) are the boundary conditions, (1.4) are the linkage conditions, (1.5) are the initial conditions, $x(t)$ denotes a k_x-dimensional vector of input functions—point perturbations, $y(t)$ is a k_y-dimensional vector of the output functions (the parameters of the motion of the absolutely rigid bodies)—the reactions of the system, $w(z, t)$ is a k_w-dimensional vector of elastic displacements of the deformable bodies, z is a k_z-dimensional vector of the space coordinates, G is a diagonal matrix of operators containing partial derivatives with respect to z and t, $n(t)$ is a k_n-dimensional vector of boundary point forces and couples at the absolutely rigid bodies (from the elastic bodies), g is a diagonal matrix of the linkage operators, F and V, P_j are k_y-dimensional and k_w-dimensional vector functions of the appropriate vector arguments, ε is a small parameter of the problem, and dots denote differentiation with respect to the time t.

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If

$$
\begin{aligned}
&-\frac{\partial F}{\partial x} = B = \text{const}, \ \frac{\partial F}{\partial y} = C = \text{const}, \ \frac{\partial F}{\partial \dot{y}} = K = \text{const}, \ \frac{\partial F}{\partial \ddot{y}} = M = \text{const} \\
&\frac{\partial F}{\partial n} = A = \text{const}, \ \frac{\partial V}{\partial y} = V_1(z), \ \frac{\partial V}{\partial \dot{y}} = V_2(z), \ \frac{\partial V}{\partial \ddot{y}} = V_3(z), \ \frac{\partial V}{\partial x} = V_4(z) \\
&\frac{\partial P_j}{\partial y} = \alpha_j = \text{const}, \ \frac{\partial F}{\partial \varepsilon} = \frac{\partial V}{\partial \varepsilon} = \frac{\partial P_j}{\partial \varepsilon} = 0; \ j = 1, \dots, 4
\end{aligned} \tag{1.6}
$$

then the linear equations of motion of the DCS are derived from (1.1)–(1.5)

$$
\begin{aligned}
&M\ddot{y} + K\dot{y} + Cy + An = Bx, \ Lw(z,t) = V_1 y + V_2 \dot{y} + V_3 \ddot{y} + V_4 x \\
&z = 0: \ w(0,t) = \alpha_1 y, \ w'(0,t)E = \alpha_2 y \\
&z = 1: \ w(1,t) = \alpha_3 y, \ y'(1,t)E = \alpha_4 y \\
&n(t) = gf(w(z,t)) \\
&t = 0: \ y(0) = y_0, \ \dot{y}(0) = \dot{y}_0, \ w(z,0) = w_0(z), \ \partial w(z,0)/\partial t = \dot{w}_0(z)
\end{aligned} \tag{1.7}
$$

where L is the diagonal matrix of a linear operator.

In what follows, as in the traditional approach [1–4], we will confine our attention to linear equations of motion of a DCS.

2. DYNAMICAL MODEL OF A LINEAR DCS

Let the vector functions $x(t), y(t), n(t)$ and $w(z, t)$ satisfy the conditions for the Laplace integral transform to exist. Setting the initial data equal to zero, we obtain from (1.7) the equations of a linear DCS in terms of the transforms (denoted by a tilde; λ is an arbitrary complex parameter)

$$(M\lambda^2 + K\lambda + C)\tilde{y}(\lambda) + A\tilde{n}(\lambda) = B\tilde{x}(\lambda) \tag{2.1}$$

$$L\tilde{w}(z,\lambda) = (V_1 + V_2\lambda + V_3\lambda^2)\tilde{y}(\lambda) + V_4\tilde{x}(\lambda) \tag{2.2}$$

$$
\begin{aligned}
&z = 0: \ \tilde{w}(0,\lambda) = \alpha_1\tilde{y}(\lambda), \ \tilde{w}'(0,\lambda)E = \alpha_2\tilde{y}(\lambda) \\
&z = 1: \ \tilde{w}(1,\lambda) = \alpha_3\tilde{y}(\lambda), \ \tilde{w}'(1,\lambda)E = \alpha_4\tilde{y}(\lambda)
\end{aligned} \tag{2.3}
$$

$$\tilde{n}(\lambda) = gf(\tilde{w}(z,\lambda)) \tag{2.4}$$

We integrate the ordinary differential equation (2.2) with respect to z with boundary conditions (2.3). Assuming that an exact solution $\tilde{w}(z, \lambda) = \tilde{w}^0[z, \lambda, \tilde{x}(\lambda), \tilde{y}(\lambda)]$ can be found for Eqs (2.2) and (2.3), and substituting this solution into (2.4), we have

$$\tilde{n}(\lambda) = \tilde{n}^0[\lambda, \tilde{x}(\lambda), \tilde{y}(\lambda)]$$

Transforming the expression for $A\tilde{n}(\lambda)$ and substituting into (2.1), we find the transform of the out vector function of the dynamical system

$$
\begin{aligned}
&\tilde{y}(\lambda) = [\Phi_{\mu\chi}(\lambda)]\tilde{x}(\lambda), \ \mu = 1, \dots, k_y; \ \chi = 1, \dots, k_x \\
&[\Phi_{\mu\chi}(\lambda)] = [(M + M_w(\lambda))\lambda^2 + (K + K_w(\lambda))\lambda + (C + C_w(\lambda))]^{-1}[B + B_w(\lambda)]
\end{aligned} \tag{2.5}
$$

where $M_w(\lambda)$, $K_w(\lambda)$, $C_w(\lambda)$, $B_w(\lambda)$ are matrices of transcendental functions of irrational expressions in λ.

Thus, we have obtained a dynamical model of a linear DCS: the $k_y \times k_x$ matrix $[\Phi_{\mu\chi}(\lambda)]$ of generalized transfer functions. In the most general case, the generalized transfer functions may be expressed as quasi-rational fractions!

$$\Phi_{\mu\chi}(\lambda) = Q_{\mu\chi}(\lambda)/D(\lambda) \tag{2.6}$$
$$D(\lambda) = \det[(M + M_w(\lambda))\lambda^2 + (K + K_w(\lambda))\lambda + (C + C_w(\lambda))] =$$
$$= A_0(\lambda)\lambda^n + A_1(\lambda)\lambda^{n-1} + \ldots + A_n(\lambda)$$
$$Q_{\mu\chi}(\lambda) = B_0(\lambda)\lambda^m + B_1(\lambda)\lambda^{m-1} + \ldots + B_m(\lambda)$$
$$\mu = 1,\ldots,k_y;\ \chi = 1,\ldots,k_x,\ n > m$$

where $Q_{\mu\chi}(\lambda)$ and $D(\lambda)$ are quasi-polynomials with variable coefficients, and n and m are positive integers. Here and below, for brevity, we will write $B_0(\lambda), B_1(\lambda), \ldots, B_m(\lambda), m$ without the subscripts $\mu\chi$.

Let us assume that $x(t)$ and $y(t)$ are absolutely integrable over the infinite interval $(-\infty, \infty)$. Then, using Fourier transforms, we have

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\Phi_{\mu\chi}(i\omega)]\tilde{x}(i\omega)e^{i\omega t}d\omega,\quad \tilde{x}(i\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

In a physically feasible system, since $x(t) = 0$ for all $t < 0$, the reaction $y(t)$ must be a real vector function of the real variable t with definition domain $[0, \infty]$. A sufficient condition for that to be true is that

$$\operatorname{Re} Q_{\mu\chi}(-i\omega) = \operatorname{Re} Q_{\mu\chi}(i\omega),\quad \operatorname{Im} Q_{\mu\chi}(-i\omega) = -\operatorname{Im} Q_{\mu\chi}(i\omega) \tag{2.7}$$
$$\operatorname{Re} D(-i\omega) = \operatorname{Re} D(i\omega),\quad \operatorname{Im} D(-i\omega) = -\operatorname{Im} D(i\omega)$$

Taking (2.7) into consideration, assuming that $x(t) = \delta(t)$ is a k_x-dimensional Dirac function (that is, $\operatorname{Re}\tilde{w}(i\omega) = \operatorname{Re}\tilde{\delta}(i\omega) = 1$, $\operatorname{Im}\tilde{w}(i\omega) = \operatorname{Im}\tilde{\delta}(i\omega) = 0$) and noting that $y(-t) = 0$, we obtain a formula for the vector of the impulse transfer functions

$$q(t) = \frac{2}{\pi}\int_0^{\infty}[\operatorname{Re}\Phi_{\mu\chi}(i\omega)]E\cos\omega t d\omega,\quad t \geqslant 0 \tag{2.8}$$

Assume

$$\lim_{|\lambda|\to\infty}\frac{D(\lambda)}{\lambda^{n+\varkappa}} = c_a = \text{const},\quad \lim_{|\lambda|\to\infty}\frac{Q_{\mu\chi}(\lambda)}{\lambda^{m+\beta}} = c_b = \text{const}$$

where $\varkappa$ and β are increments in the degrees of the respective quasi-polynomials $D(\lambda)$ and $Q_{\mu\chi}(\lambda)$ as $|\lambda| \to \infty$.

Since $q(0) = 0$ in a physically feasible DCS, it follows from a well-known formula of the operational calculus for the initial value of the inverse transform that necessarily

$$\lim_{t\to 0} q(t) = \lim_{|\lambda|\to\infty}\lambda\tilde{q}(\lambda) = \lim_{|\lambda|\to\infty}\lambda^{m+\beta+1-n-\varkappa}[(Q_{\mu\gamma}(\lambda)/\lambda^{m+\beta})/(D(\lambda)/\lambda^{n+\varkappa})]E = 0$$

Consequently

$$n + \varkappa > m + \beta + 1,\quad |c_b| < \infty,\quad c_a \neq 0 \tag{2.9}$$

We will call the linear dynamical model of the DCS (2.6) physically feasible if conditions (2.7) and (2.9) are satisfied.

3. THE STABILITY OF DYNAMICAL MODELS OF DCS

To investigate the stability of physically feasible dynamical models of DCS, we introduce the following definitions.

Definition 1. A dynamical system is stable if

$$|q(t)| < \infty \text{ for all } t \geqslant 0 \tag{3.1}$$

Definition 2. A dynamical system is asymptotically stable if condition (3.1) holds and $\lim|q(t)| = 0$ as $t \to \infty$.

Theorem 1. Let the variable coefficients $A_j(\lambda)$ ($j = 0, \dots, n$) of the quasi-polynomial $D(\lambda)$ be analytic functions in the right complex half-plane $\lambda = \alpha + i\omega$ and assume that

$$\forall\alpha > 0: \lim_{|\lambda|\to\infty} (D(\lambda)/\lambda^{n+\varkappa}) = c_a \neq 0 \tag{3.2}$$

$$\forall\omega \in (-\infty,\infty): \quad D(i\omega) = u(\omega) + iv(\omega) \neq 0, \quad u(-\omega) = u(\omega), \quad v(-\omega) = -v(\omega)$$

Suppose that as ω increases monotonically from 0 to ∞, the vector $D(i\omega)$ rotates in the complex plane (u, iv) from the positive real axis u in the positive direction through an angle $(n + \varkappa)\pi/2$. Then all the roots of the quasi-polynomial $D(\lambda)$ lie to the left of the imaginary axis of the complex λ plane.

Proof. As in the case of systems with lumped parameters [5], the proof relies on the argument principle [6]. In the complex plane of $\lambda = \alpha + i\omega$, let Ω be an open domain bounded by a closed contour $S = S_1 + S_2$, where S_1 is a semicircle of radius r about the point (0, 0) in the right half-plane and S_2 is the interval $[-ir, ir]$. By assumption, the function $D(\lambda)$ is analytic in Ω, and as $r \to \infty$ it has no zeros on the contour S. By the argument increment principle, as the point λ describes one complete circuit around S in the counterclockwise sense, the argument of the function $D(\lambda)$ receives an increment

$$\lim_{r=|\lambda|\to\infty} \Delta_s \arg D(\lambda) = \lim_{r=|\lambda|\to\infty} \Delta_{s_1} \arg D(\lambda) + \lim_{r=|\lambda|\to\infty} \Delta_{s_2} \arg D(\lambda) = 2\pi N \tag{3.3}$$

where N is the number of zeros of $D(\lambda)$ in the right half-plane. Taking the first condition of (3.2) into consideration, we have

$$\lim_{r=|\lambda|\to\infty} \Delta_{s_1} \arg D(\lambda) = (n + \varkappa)\pi \tag{3.4}$$

Since $\forall\lambda \in S_2 : \lambda = i\omega$, it follows, by the remaining conditions of (3.2), that

$$\lim_{r\to\infty} \Delta_{s_2} \arg D(\lambda) = \lim_{r\to\infty} (\Delta_{r\geq\omega\geq-r} \arg D(i\omega)) = -2\Delta_{0\leq\omega<\infty} \arg D(i\omega) \tag{3.5}$$

Substituting expressions (3.4) and (3.5) into (3.3) and putting $N = 0$, we obtain a condition for the zeros of $D(\lambda)$ in the complex λ plane to be located to the left of the imaginary axis

$$\Delta_{0\leq\omega<\infty} \arg D(i\omega) = (n + \varkappa)\pi/2$$

which is was required to prove.

Theorem 2. Suppose the quasi-polynomials $Q_{\mu\chi}(\lambda)$ in the physically feasible quasi-rational fractions (2.6) are analytic in the open complex plane and each function $\Phi_{\mu\chi}(\lambda)$has a denumerable set of isolated poles $\lambda = \lambda_\nu = \alpha_\nu + i\omega_\nu$ ($\nu = 1, 2, \dots$), among which there is one pole $\lambda = \lambda_1$ of multiplicity k and all the other poles are simple.

Then: (1) if the dynamical model (2.6) satisfies Theorem 1, it is asymptotically stable; (2) if one root of the generalized determinant $D(\lambda)$ is zero, then the dynamical model is stable but not asymptotically stable; (3) if two or more roots are zero, the dynamical model is unstable.

Proof. In the λ plane, let H be an open domain bounded by a closed contour $\eta = \eta_1 + \eta_2$, where the straight-line segment $\eta_1 = [\alpha_0 - ir, \alpha_0 + ir]$ passes through the point $(\alpha_0, 0)$, $\alpha_0 > 0$, and is parallel to the imaginary axis, while η_2 is a semicircle of radius r about $(\alpha_0, 0)$, to the left of the straight line $\alpha = \alpha_0$, $\alpha_0 \ll r$.

The vector of impulse transfer functions of the dynamical model may be represented as follows, using Mellin's formula [5]:

$$q(t) = \frac{1}{2\pi i} \int_{\alpha_0 - i_\infty}^{\alpha_0 + i_\infty} e^{\lambda t} [\Phi_{\mu\chi}(\lambda)] E d\lambda, \quad \alpha_0 > 0, \quad t \geq 0 \tag{3.6}$$

The poles λ_ν ($\nu = 1,2,\dots$) of the functions $\Phi_{\mu\chi}(\lambda)$ are isolated; they are also poles of the functions $e^{\lambda t}\Phi_{\mu\chi}(\lambda)$ and as $r \to \infty$ they belong to H. By the Residue theorem

$$q(t) = \frac{1}{2\pi i}\left(\lim_{r\to\infty}\int_{\alpha_0-ir}^{\alpha_0+ir} e^{\lambda t}[\Phi_{\mu\chi}(\lambda)]Ed\lambda + \lim_{r\to\infty}\int_{\eta_2} e^{\lambda t}[\Phi_{\mu\chi}(\lambda)]Ed\lambda\right) =$$

$$= \operatorname*{Res}_{\lambda=\lambda_1}(e^{\lambda t}[\Phi_{\mu\chi}(\lambda)]E) + \sum_{j=2}^{\infty}\operatorname*{Res}_{\lambda=\lambda_j}(e^{\lambda t}[\Phi_{\mu\chi}(\lambda)]E) \tag{3.7}$$

Indeed, for physically feasible dynamic models, by relations (2.9), we have $\vartheta = n + \varkappa - m - \beta > 1$, and a real number $M > 0$ exists, such that, for $\infty > t \geqslant 0$,

$$\forall\lambda \in \eta_2:\ |e^{\lambda t}\Phi_{\mu\chi}(\lambda)| \leqslant M/|\lambda|^{\vartheta} \text{ for } |\lambda| \geqslant r \to \infty;\ \ \vartheta > 1$$

Consequently

$$\int_{\eta_2} e^{\lambda t}\Phi_{\mu\chi}(\lambda)d\lambda \leqslant \frac{M\pi}{r^{\vartheta-1}} \to 0 \text{ as } r \to \infty$$

Letting in $r \to \infty$ in (3.7) and using (3.6), we conclude that the last equality in (3.7) is true.
Since the pole λ_1 is of multiplicity k, while the other poles $\lambda_2, \lambda_2, \ldots$ are simple, we have

$$q(t) = \frac{1}{(k-1)!}\lim_{\lambda\to\lambda_1}\frac{d^{k-1}}{d\lambda^{k-1}}\left(e^{\lambda t}(\lambda-\lambda_1)^k\frac{[Q_{\mu\chi}(\lambda)]}{D(\lambda)}E\right) +$$

$$+\sum_{j=2}^{\infty}\lim_{\lambda\to\lambda_j}\left(e^{\lambda t}(\lambda-\lambda_j)\frac{[Q_{\mu\chi}(\lambda)]}{D(\lambda)}E\right) =$$

$$= e^{\lambda_1 t}\sum_{\xi=1}^{k} t^{k-\xi}\left(\frac{[Q_{\mu\chi}(\lambda)]}{\Psi(\lambda)}\right)^{\xi-1}\Bigg|_{\lambda=\lambda_1}\frac{E}{(\xi-1)!(k-\xi)!} + \sum_{j=2}^{\infty} e^{\lambda_j t}\lambda_j^{m-n}\frac{[Q_{\mu\chi}(\lambda_j)]}{\lambda_j^m}\frac{\lambda_j^n}{D'(\lambda_j)}E \tag{3.8}$$

$$n > m,\ \ \Psi(\lambda) = \sum_{i=0}^{\infty}\frac{(\lambda-\lambda_1)^i D^{(k+i)}(\lambda_1)}{(k+i)!}$$

Since $\Psi(\lambda_1) \neq 0$, $D'(\lambda_j) \neq 0$ and $Q_{\mu\chi}(\lambda)$ are analytic functions in the open complex plane, it follows that a real number $C > 0$ exists such that

$$\left|\left(\frac{Q_{\mu\chi}(\lambda)}{\Psi(\lambda)}\right)^{\xi-1}\Bigg|_{\lambda=\lambda_1}\right| < C,\ \ \xi = 1,2,\ldots k,\ \ \left|\frac{Q_{\mu\chi}(\lambda_j)}{\lambda_j^m}\frac{\lambda_j^n}{D'(\lambda_j)}\right| < C,\ \ j = 2,3,\ldots \tag{3.9}$$

Note that, since the functions $Q_{\mu\chi}(\lambda)$ are analytic, the poles $\lambda = \lambda_1, \lambda_2, \ldots$ of the functions $\Phi_{\mu\chi}(\lambda) = Q_{\mu\chi}(\lambda) / D(\lambda)$ are zeros of $D(\lambda)$.
Let the quasi-polynomial $D(\lambda)$ satisfy Theorem 1, that is, $\mathrm{Re}\lambda_\nu = \alpha_\nu < 0$, $\max\{\alpha_\nu\} = -\alpha$, $\alpha > 0$ ($\nu = 1,2,\ldots$). Then it follows from (3.8) and (3.9) that

$$|q(t)| < [C]e^{-\alpha t}\left(\sum_{\xi=1}^{k}\frac{t^{k-\xi}}{(k-\xi)!(\xi-1)!} + \Sigma\right),\ \ t \geqslant 0$$

$$[C] = [C,\ldots,C]^T,\ \ \Sigma = \sum_{j=2}^{\infty}|\lambda_j|^{m-n}$$

We designate a subscript j such that $|\lambda_{j+1}| > |\lambda_j|$. Then, by the integral criterion, the series Σ is convergent. In addition

$$\lim e^{-\alpha t} = 0,\ \ \lim t^{k-\xi}e^{-\alpha t} = 0 \text{ as } t \to \infty$$

Consequently, $\forall t \geqslant 0: |q(t)| < \infty$, $\lim |q(t)| = 0$ as $t \to \infty$, and by Definition 2 the dynamical model is asymptotically stable.
Suppose $k = 1$ and $\lambda_1 = 0$, in which case $D(\lambda) = \lambda D_1(\lambda)$, $D_1(0) \neq 0$. Taking relations (3.8) and (3.9)

into consideration, we have

$$|q(t)| \leqslant [|Q_{\mu\chi}(0)|/|D_1(0)|]E + [C]e^{-\alpha t}\Sigma, \quad t \geqslant 0$$

Hence it follows that $|q(t)| < \infty$, and by (3.8),

$$\lim |q(t)| = [|Q_{\mu\chi}(0)|/|D_1(0)|] \neq 0 \quad \text{as} \quad t \to \infty$$

that is, the dynamical model is stable but not asymptotically stable.

Suppose $k = 2$ and $\lambda_1 = 0$. By (3.8), we find that

$$q(t) = t\left[\frac{Q_{\mu\chi}(0)}{\Psi(0)}\right]E + \left[\frac{Q_{\mu\chi}(\lambda)}{\Psi(\lambda)}\right]'\Bigg|_{\lambda=0} E + \sum_{j=2}^{\infty} e^{\lambda_j t}\left[\frac{Q_{\mu\chi}(\lambda_j)}{D'(\lambda_j)}\right]E$$

It is obvious that $|q(t)| \to \infty$ as $t \to \infty$, and the dynamical model is unstable if two roots vanish. It can be shown similarly that if $k \geqslant 3$ and $\lambda_1 = 0$, the dynamical model is unstable.

Remark. Suppose that among the poles $\lambda = \lambda_\nu$ ($\nu = 1,2\ldots$) of the functions $\Phi_{\mu\chi}(\lambda)$, which are zeros of the quasi-polynomial $D(\lambda)$, there is at least one pole $\lambda = \lambda_p$ in the right half of the complex λ plane, that is, Re $\lambda_p = \alpha_p >$ 0. Then the dynamical model (2.6) is unstable, since, in view of the fact that

$$|\exp(\lambda_p t)| = \exp(\alpha_p t) \to \infty \quad \text{as} \quad t \to \infty$$

it follows from (3.8), taking into account inequality (3.9), that $|q(t)| \to \infty$ as $t \to \infty$.

4. QUASI-STATICAL MODEL OF A LINEAR DCS

We will construct a quasi-statical model of a linear DCS by the method of successive approximations. At the first stage, we exclude all terms in the partial differential equations (PDEs) of system (1.7) that contain derivatives with respect to the time t, integrate with respect to the spatial coordinates z and, for the given boundary conditions, find a statical solution of the PDEs to serve as the zeroth approximation. At the second stage, we substitute the solution of the zeroth approximation into the terms of the PDEs that contain partial derivatives with respect to time, also including the terms $V_2\dot{y}$ and $V_3\ddot{y}$ on the right of the PDE in (1.7) and, again integrating with respect to z with zero boundary conditions, find a quasi-statical solution of the PDEs, which depends on time through the boundary conditions and the right-hand sides of the PDEs. Substituting this quasi-statical solution into the linkage conditions and then into the ordinary differential equations (ODEs) of system (1.7), we obtain a quasi-statical model of the linear DCS as an ordinary dynamical system, involving only ODEs and initial conditions. Putting the initial data equal to zero and changing to Laplace transforms, we find that

$$\begin{aligned}
&\tilde{y}(\lambda) = [\overline{\Phi}_{\mu\chi}(\lambda)]\tilde{x}(\lambda), \quad \mu = 1,\ldots,k_y; \ \chi = 1,\ldots,k_x \\
&[\overline{\Phi}_{\mu\chi}(\lambda)] = [(M+\overline{M}_w)\lambda^2 + (K+\overline{K}_w)\lambda + (C+\overline{C}_w)]^{-1}(B+\overline{B}_w) \\
&\overline{\Phi}_{\mu\chi}(\lambda) = \overline{Q}_{\mu\chi}(\lambda)/\overline{D}(\lambda), \ \overline{Q}_{\mu\chi}(\lambda) = B_0\lambda^m + B_1\lambda^{m-1} + \ldots + B_m \\
&\overline{D}(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \ldots + A_n, \quad n > m
\end{aligned} \tag{4.1}$$

where $\overline{M}_w$, $\overline{K}_w$, $\overline{C}_w$, $\overline{B}_w$ are the matrices of the added constant coefficients of inertia, damping, elastic stiffness and perturbation, respectively, due to the impact of the elastic bodies or the absolutely rigid bodies; $\overline{Q}_{\mu\chi}(\lambda)$ and $\overline{D}(\lambda)$ are ordinary polynomials with constant coefficients.

Note that the quasi-statical model (4.1) follows from the dynamical model (2.5), (2.6) if one expands $M_w(\lambda)$, $K_w(\lambda)$, $C_w(\lambda)$ and $B_w(\lambda)$ in powers of λ, substitutes into (2.5) and (2.6) and retains terms containing λ up to and including the second power. If we set

$$\overline{M}_w = \lim M_w(\lambda), \ \overline{K}_w = \lim K_w(\lambda), \ \overline{C}_w = \lim \overline{C}_w(\lambda), \ \overline{B}_w = \lim \overline{B}_w(\lambda) \quad \text{as} \quad \lambda \to 0$$

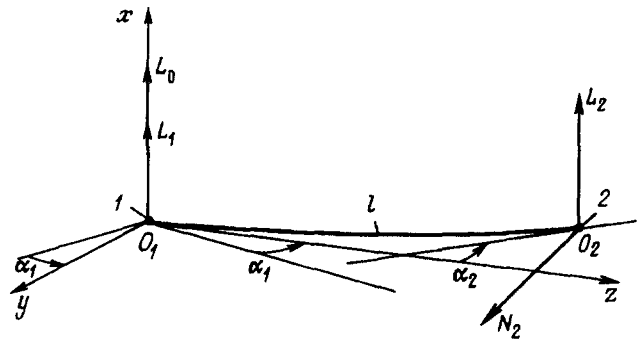

Fig. 1.

in (4.1), the result is a limiting quasi-statical model of the DCS. Like quasi-statical models, mathematical models of deformable control objects whose construction is based on an approximate solution of PDEs by the finite-element method, or by expansion in the first characteristic forms, are finite-dimensional. A control device designed on the basis of such approximate models may cause destabilization of the neglected modes of vibration. This effect is referred to in [7], and has been observed experimentally in large space structures; it is called control spillover [8].

5. AN EXAMPLE OF THE DYNAMICAL MODELLING OF THE ELASTIC LIMB OF A MANIPULATOR

As applied to manipulators, we will investigate the problem of controlling the plane angular motion of a discrete-continuous mechanical model (Fig. 1) with an elastic homogeneous straight rod of length l in which internal friction is taken into account using the Voigt model. The beginning of the rod is rigidly attached at the point O_1 of an absolutely rigid beam 1 with moment of inertia J_0^*. An absolutely rigid body 2 of mass m_2^* and moment of inertia J_2^* is attached at its centre of mass O_2 to the other end of the rod. A control torque

$$L_0^* = p^*\alpha_0^* - p^*\Pi\alpha_1^* - k_0^*\dot{\alpha}_1^*$$

is applied to the beam 1, where α_0^* is the control (input function), α_1^* is the angle of rotation of the beam, Π is the operator of a correcting device, p^* is the amplification factor and k_0^* is the damping factor. Taking into account the fact that the deformation of the rod is small, we can write the following expression for the polar angle of the output point O_2 of the flexible rod

$$\alpha^* = \alpha_1^* - y_1^*/l, \quad |y^*(l,t)| = |y_1^*(t)| \ll l$$

Letting L_1^*, L_2^*, N_2^* denote the reactions of the rod at its ends, we write the dimensional equations of motion of this DCS as

$$J_0^*\ddot{\alpha}_1^* + k_0^*\dot{\alpha}_1^* + p^*\Pi\alpha_1^* = p^*\alpha_0^* + L_1^*, \quad \alpha^* = \alpha_1^* - y_1^*/l, \quad J_2^*(\ddot{\alpha}_1^* + \ddot{\alpha}_2^*) = L_2^*$$

$$m_2^*\ddot{y}_1^* - m_2^* l\ddot{\alpha}_1^* = N_2^*$$

$$\rho y_{tt}^* + EJ(1 + h\partial/\partial t^*)y_{zzzz}^* = \rho z^*\ddot{\alpha}_1^*$$

$$y^*(0,t^*) = 0, \quad y_z^*(0,t^*) = 0, \quad y^*(l,t^*) = y_1^*(t^*), \quad y_z^*(l,t^*) = -\alpha_2^*(t^*)$$

$$L_1^* = -EJ(1 + h\partial/\partial t^*)y_{zz}^*(0,t^*), \quad L_2^* = EJ(1 + h\partial/\partial t^*)y_{zz}^*(l,t^*)$$

$$N_2^* = EJ(1 + h\partial/\partial t^*)y_{zzz}^*(l,t^*)$$

$$t^* = 0\colon\ \alpha_1^*(0) = 0, \quad \dot{\alpha}_1^*(0) = 0, \quad \alpha_2^*(0) = 0, \quad \dot{\alpha}_2^*(0) = 0$$

$$y_1^*(0) = 0, \quad \dot{y}_1^*(0) = 0, \quad y^*(z^*,0) = 0, \quad y_t^*(z^*,0) = 0$$

where ρ is the density per unit length of the rod, E and J are the Young's modulus and the equatorial moment of inertia of the cross-section, h is the Voigt coefficient of internal friction, $y^*(z^*, t^*)$ is the deflection of the rod, $\alpha_0^*(t^*)$ is the input function and $\alpha^*(t^*)$ is the output function.

Introducing the dimensionless variables and parameters

$$t = \frac{t^*}{T},\ T = \left(\frac{\rho l^4}{EJ}\right)^{1/2},\ y = \frac{y^*}{\delta},\ y_1 = \frac{y_1^*}{\delta},\ z = \frac{z^*}{l},\ \frac{\delta}{l} \ll 1,\ \gamma = \frac{h}{T}$$

$$\alpha_r = \frac{l}{\delta}\alpha_r^*,\ r = 0,1,2;\ p = \frac{l}{EJ}p^*,\ J_0 = \frac{J_0^*}{\rho l^3},\ J_2 = \frac{J_2^*}{\rho l^3},\ m_2 = \frac{m_2^*}{\rho l},\ k_0 = \frac{l}{EJT}k_0^*$$

$$L_1 = \frac{L_1^*}{L},\ L_2 = \frac{L_2^*}{L},\ L = \frac{EJ\delta}{l^2},\ N_2 = \frac{N_2^*}{N},\ N = \frac{EJ\delta}{l^3}$$

we write the equations of motion of the system in dimensionless form and, changing to Laplace transforms, we obtain equations for the linear DCS in terms of transforms

$$\begin{aligned}&(J_0\lambda^2 + k_0\lambda + p\Pi(\lambda))\alpha_1(\lambda) - L_1(\lambda) = p\alpha_0(\lambda),\quad \Pi(\lambda) = A(\lambda)/B(\lambda)\\ &J_2\lambda^2(\alpha_1(\lambda) + \alpha_2(\lambda)) - L_2(\lambda) = 0\end{aligned} \tag{5.1}$$

$$\begin{aligned}&m_2\lambda^2(y_1(\lambda) - \alpha_1(\lambda)) - N_2(\lambda) = 0,\quad \alpha(\lambda) = \alpha_1(\lambda) - y_1(\lambda)\\ &y_{zzzz}(z,\lambda) - n^4 y(z,\lambda) = -n^4 z\alpha_1(\lambda),\quad n^4 = -\lambda^2/(1+\gamma\lambda)\end{aligned} \tag{5.2}$$

$$y(0,\lambda) = y_z(0,\lambda) = 0,\quad y(1,\lambda) = y_1(\lambda),\quad y_z(1,\lambda) = -\alpha_2(\lambda) \tag{5.3}$$

$$\begin{aligned}&L_1(\lambda) = -(1+\gamma\lambda)y_{zz}(0,\lambda),\quad L_2(\lambda) = (1+\gamma\lambda)y_{zz}(1,\lambda)\\ &N_2(\lambda) = (1+\gamma\lambda)y_{zzz}(1,\lambda)\end{aligned} \tag{5.4}$$

where $\Pi(\lambda)$ is a proper rational fraction.

The general solution of inhomogeneous ordinary differential equation (5.2) may be written as

$$\begin{aligned}&y(z,\lambda) = C_1 S(nz) + C_2 T(nz) + C_3 U(nz) + C_4 V(nz) + z\alpha_1(\lambda)\\ &S(x) = (\operatorname{ch} x + \cos x)/2,\quad T(x) = (\operatorname{sh} x + \sin x)/2,\quad U(x) = (\operatorname{ch} x - \cos x)/2,\\ &V(x) = (\operatorname{sh} x - \sin x)/2\end{aligned}$$

where C_1, C_2, C_3, C_4, are integration constants determined from boundary conditions (5.3).

Substituting the expression $y(z, \lambda)$ into (5.4) and then into Eq. (5.4), we obtain

$$\alpha(\lambda) = (Q(\lambda)/D(\lambda))\alpha_0(\lambda),\ \alpha_1(\lambda) = (Q_1(\lambda)/D(\lambda))\alpha_0(\lambda) \tag{5.5}$$

where

$$Q(\lambda) = \begin{vmatrix}\varphi_{11} & \varphi_{12}+\varphi_{13} & B(\lambda)p\\ \varphi_{21} & \varphi_{22}+\varphi_{23} & 0\\ \varphi_{31} & \varphi_{32}+\varphi_{33} & 0\end{vmatrix},\quad Q_1(\lambda) = \begin{vmatrix}\varphi_{11} & \varphi_{12} & B(\lambda)p\\ \varphi_{21} & \varphi_{22} & 0\\ \varphi_{31} & \varphi_{32} & 0\end{vmatrix}$$

$$\begin{aligned}&D(\lambda) = \det[\varphi_{ij}],\quad i,j = 1,2,3\\ &\varphi_{11} = B(\lambda)\mu_{11}(1+\gamma\lambda),\quad \varphi_{12} = B(\lambda)\mu_{12}(1+\gamma\lambda)\\ &\varphi_{13} = B(\lambda)(J_0\lambda^2 + k_0\lambda + \mu_{13}\lambda^2) + A(\lambda)p\\ &\varphi_{21} = J_2\lambda^2 + \mu_{21}(1+\gamma\lambda),\quad \varphi_{22} = \mu_{22}(1+\gamma\lambda),\quad \varphi_{23} = J_2\lambda^2 - \mu_{23}\lambda^2\\ &\varphi_{31} = \mu_{31}(1+\gamma\lambda),\quad \varphi_{32} = m_2\lambda^2 + \mu_{32}(1+\gamma\lambda),\quad \varphi_{33} = -m_2\lambda^2 - \mu_{33}\lambda^2\\ &\mu_{11} = f_{11}n,\quad \mu_{12} = f_{12}n^2,\quad \mu_{13} = f_{13}n^{-3},\quad \mu_{21} = f_{21}n,\quad \mu_{22} = f_{22}n^2,\quad \mu_{23} = f_{23}n^{-3}\\ &\mu_{31} = f_{31}n^2,\quad \mu_{32} = f_{32}n^3,\quad \mu_{33} = f_{33}n^{-2}\\ &f_{11} = V/\Delta,\quad f_{12} = U/\Delta,\quad f_{13} = [U(n-T) - V(1-S)]/\Delta,\quad f_{21} = (UT - VS)/\Delta\\ &f_{22} = (T^2 - US)/\Delta,\quad f_{23} = V - [VS(1-S) - US(n-T) + T^2(n-T) - UT(1-S)]/\Delta\\ &f_{31} = (US - V^2)/\Delta,\quad f_{32} = (TS - UV)/\Delta\end{aligned}$$

$$\mu_{ij}(1+\gamma\lambda) = a_{ij}^{(0)}(1+\gamma\lambda) + a_{ij}^{(2)}\lambda^2 + O(|\lambda|^3), \quad i =, 1, 2, 3; \quad j = 1, 2$$
$$\mu_{ij}\lambda^2 = a_{ij}^{(0)}\lambda^2 + O(|\lambda|^3), \quad i = 1, 2, 3; \; j = 3$$

Formulae (5.5) define a DCS with a dynamical model of the rod, where the whole infinite spectrum of eigenfrequencies and vibration modes of the rod is taken into account through the variable coefficients μ_{ij} $(i, j = 1, 2, 3)$. Note that

$$\begin{aligned}
&\mu_{ij} = a_{ij}^{(0)} + \lambda^2 a_{ij}^{(2)} + O(|\lambda|^3) \;\text{ for }\; |\lambda| \ll 1 \\
&a_{11}^{(0)} = 2, \quad a_{11}^{(2)} = -1/140, \quad a_{12}^{(0)} = 6, \quad a_{12}^{(2)} = -13/420, \quad a_{13}^{(0)} = 1/30 \\
&a_{21}^{(0)} = 4, \quad a_{21}^{(2)} = 1/105, \quad a_{22}^{(0)} = 6, \quad a_{22}^{(2)} = 11/210, \quad a_{23}^{(0)} = 1/20 \\
&a_{31}^{(0)} = 6, \quad a_{31}^{(2)} = 11/210, \quad a_{32}^{(0)} = 12, \quad a_{32}^{(2)} = 13/35, \quad a_{33}^{(0)} = 7/20
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
&\mu_{ij} \to b_{ij}\lambda^{k_{ij}} \;\text{ as }\; |\lambda| \to \infty \\
&k_{11} = k_{12} = 0; \quad k_{13} = -k_{32} = -0.75; \quad k_{21} = -k_{33} = 0.25; \quad k_{22} = -k_{23} = k_{31} = 0.5
\end{aligned} \tag{5.7}$$

Substituting formulae (5.6) into relations (5.5) and noting that

$$\mu_{ij}(1+\gamma\lambda) = a_{ij}^{(0)}(1+\gamma\lambda) + a_{ij}^{(2)}\lambda^2 + O(|\lambda|^3), \quad i = 1, 2, 3; \quad j = 1, 2$$
$$\mu_{ij}\lambda^2 = a_{ij}^{(0)}\lambda^2 + O(|\lambda|^3), \quad i = 1, 2, 3; \; j = 3$$

we obtain a model of the elastic link of the manipulator with a quasi-statical model of the rod.

Assuming that $\mu_{ij} = \alpha_{ij}^{(0)}$ $(i, j = 1, 2, 3)$, we have a model of the elastic link of the manipulator with a limiting quasi-statical model of the rod.

Model (5.5) is convenient for parametric design. Let $\alpha_0(t) = \delta(t)$ be a Dirac function. Then, by formula (2.8), the impulse transfer function with respect to the output $\alpha(t)$ is

$$q(t) = \frac{2}{\pi}\int_0^\infty \operatorname{Re}\Phi(i\omega)\cos\omega t\, d\omega, \quad \Phi(i\omega) = \frac{Q(i\omega)}{D(i\omega)}$$

If $\alpha_0(t) = 1(t)$ is the Heaviside unit step function, then the transfer function of the system with respect to the output $\alpha(t)$ may be expressed as

$$\alpha(t) = \int_0^t q(t)\,dt = \frac{2}{\pi}\int_0^\infty \operatorname{Re}\Phi(i\omega)\frac{\sin\omega t}{\omega}\,d\omega, \quad t \geqslant 0 \tag{5.8}$$

The transfer function with respect to the angle of rotation $\alpha_1(t)$ of beam 1 may be evaluated in a similar way.

Let us select the desired frequency response of the system

$$\operatorname{Re}\Phi\;(i\omega) = (1-\tau^2\omega^2)/((1-\tau^2\omega^2)^2 + 2\tau^2\omega^2)$$

to which corresponds, according to (5.8), the desired transfer function $\alpha^0(t)$, which is exponential in nature and the duration of the transient does not exceed 3τ.

Let $\operatorname{Re}\Phi(i\omega)$ depend on the design parameters $T_1, T_2 \ldots, T_s$ with constraints $T_j \in [a_j, b_j]$. Consider the integral

$$\int_0^\infty [\operatorname{Re}\Phi(i\omega, T_1, T_2, \ldots, T_s) - \operatorname{Re}\Phi^\circ(i\omega)]^2\, d\omega \tag{5.9}$$

Carrying out multi-dimensional minimization, we choose $T_1^\circ, T_2^\circ \ldots, T_s^\circ$ for which integral (5.9) reaches a minimum. We can now compute the transfer function of the system being designed

$$\alpha_c(t) = \frac{2}{\pi}\int_0^\infty \operatorname{Re}\Phi(i\omega, T_1^0, T_2^0, \ldots, T_s^0)\frac{\sin\omega t}{\omega}\,d\omega$$

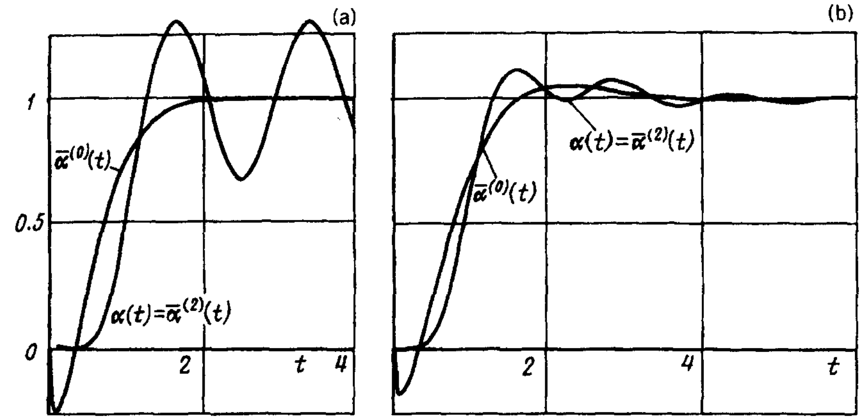

Fig. 2.

and analyse it. The degree of the approximation of $\alpha_c(t)$ to $\alpha^\circ(t)$ depends on the structure of the control device and the correcting links.

Consider the model of a DCS without a load at the end of the rod, $m_2 = J_2 = 0$, with parameters $J_0 = 7.1 \times 10^{-3}$; $k_0 = 1.5 \times 10^{-2}$; $\gamma = 4.8 \times 10^{-3}$; $p = 31.5$ and correcting device

$$\Pi_0(\lambda) = \frac{(T_1\lambda+1)(T_2\lambda+1)(T_3\lambda+1)}{(T_4\lambda+1)(T_5\lambda+1)(T_6\lambda+1)} \tag{5.10}$$

First, on the assumption of the limiting quasi-statical model of the rod ($\mu_{ij} = a_{ij}^{(0)}, i, j = 1, 2, 3$), the parameters of the correcting device were chosen by minimizing integral (5.9)

$$T_1 = 0.36;\quad T_2 = 0.2544;\quad T_3 = 0.246;\quad T_4 = 3.8 \times 10^{-2};\quad T_5 = 10^{-2};\quad T_6 = 4.76 \times 10^{-6} \tag{5.11}$$

and the transfer function $\bar{\alpha}^{(0)}(t)$ was computed.

The limiting quasi-statical model of the rod in this system was then replaced by a dynamical model and the transfer function $\alpha(t)$ was computed. A similar computation yielded the transfer function $\bar{\alpha}^{(2)}(t)$ of this system with the quasi-statical rod model (5.6). The result is shown in Fig. 2(a). It can be seen that the correcting device $\Pi_0(\lambda)$ constructed on the basis of the limiting quasi-statical model of the rod destabilizes the system, with both dynamical and quasi-statical models of the rod, at a neglected vibration mode. Note that the graphs of $\alpha(t)$ and $\bar{\alpha}^{(2)}(t)$ are practically the same. Figure 2(b) shows graphs of $\alpha(t)$ and $\bar{\alpha}^{(0)}(t)$, $\bar{\alpha}^{(2)}(t)$ for $p = 0.315$, from which it can be seen that reducing the amplification factor

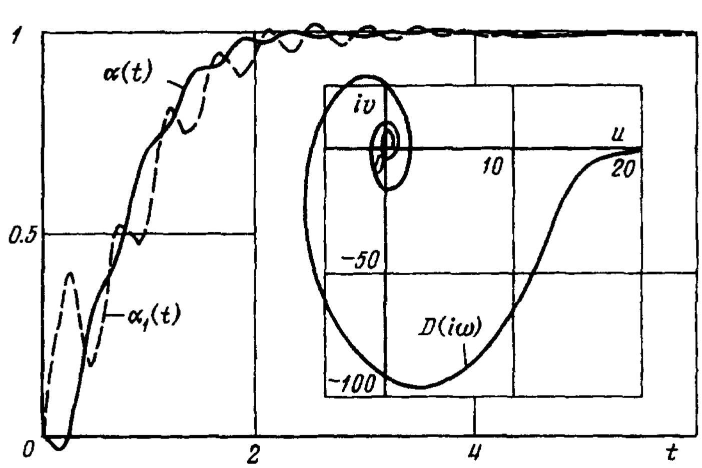

Fig. 3.

p lessens the difference between the results of limiting quasi-statical modelling $\bar{\alpha}^{(0)}(t)$, quasi-statical modelling $\bar{\alpha}^{(2)}(t)$ and dynamical modelling $\alpha(t)$.

Next, a correcting device was constructed for the DCS with a dynamical model of the rod, where $m_2 = J_2 = 0, p = 31.5$

$$\Pi(\lambda) = \Pi_0(\lambda)\frac{(T_7^2\lambda^2 + 2\xi_1 T_7\lambda + 1)(T_8^2\lambda^2 + 2\xi_2 T_8\lambda + 1)}{(T_9^2\lambda^2 + 2\xi_3 T_9\lambda + 1)(T_{10}^2\lambda^2 + 2\xi_4 T_{10}\lambda + 1)} \tag{5.12}$$

where

$$T_7 = 7.246 \times 10^{-2};\quad \xi_1 = 10^{-3};\quad T_8 = 2.934 \times 10^{-2};\quad \xi_2 = 7 \times 10^{-2}$$

$$T_9 = 0.284;\quad \xi_1 = 10^{-3};\quad T_{10} = 4.575 \times 10^{-2};\quad \xi_4 = 4 \times 10^{-2}$$

and the function $\Pi_0(\lambda)$ is given by (5.10) and (5.11).

Figure 3 shows graphs of the transfer functions $\alpha(t)$ and $\alpha_1(t)$, as well as the frequency hodograph $D(i\omega)$ of the system in the special scale

$$u + iv = D(i\omega)(\text{Arsh}\,|D(i\omega)|)/|D(i\omega)|$$

The system satisfies Theorem 1, and here $n + \varkappa = 12$ and $\varkappa = 1$.

In the case of a system with $p = 31.5$ and a load $m_2 = 0.34$ and $J_2 = 0.1$ at the end of the rod, with the design based on a limiting quasi-statical model of the rod, the correcting device has the form (5.12), where the function $\Pi_0(\lambda)$ is given by (5.10) with

$$T_1 = 1.36;\quad T_2 = 0.96;\quad T_3 = 4 \times 10^{-3};\quad T_3 = 3.81 \times 10^{-2};\quad T_5 = 10^{-2};\quad T_6 = 10^{-4}$$

$$T_7 = 0.2206;\quad \xi_1 = 10^{-2};\quad T_8 = 5.54 \times 10^{-2};\quad \xi_4 = 5 \times 10^{-2}$$

$$T_9 = 0.446;\quad \xi_1 = 4 \times 10^{-3};\quad T_{10} = 0.1193;\quad \xi_4 = 10^{-2}$$

Graphs of the transfer functions of a system with this correcting device, where the models of the rod are dynamical $\alpha(t)$, quasi-statical $\bar{\alpha}^{(2)}(t)$ and limiting quasi-statical $\bar{\alpha}^{(0)}(t)$, are shown in Fig. 4. A comparison of the graphs in Figs 4 and 2(a) shows that the introduction of a load at the end of the rod reduces the difference between the graphs $\alpha(t) = \bar{\alpha}^{(2)}(t)$ and $\bar{\alpha}^{(0)}(t)$ to within an error that is acceptable in practice.

Considering the above systems with a dynamical model of the rod and letting the internal friction in the rod decrease, $\gamma \to 0$, values of γ were found for which it was not possible to formulate a finite-dimensional correcting device in the form of a rational fraction so that the frequency hodograph $D(i\omega)$ satisfies Theorem 1, i.e., so that the control system is stable; this is in agreement with previously known results [9]. However, in a system with a limiting quasi-statical model of the rod, even if $\gamma = 0$ one can construct a finite-dimensional correcting device which guarantees stability of the control system and the prescribed quality of the transfer function $\bar{\alpha}^{(0)}(t)$.

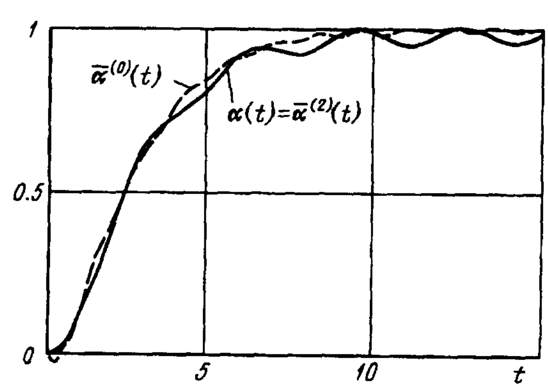

Fig. 4.

REFERENCES

1. AKULENKO, L. D., Quasi-stationary finite control of the motion of hybrid oscillatory systems. *Prikl. Mat. Mekh.*, 1991, **55**, 2, 183–192.
2. LAVROVSKII, E. K. and FORMAL'SKII, A. M., Control of an elastic manipulator arm using load position and velocity feedback. *Prikl. Mat. Mekh.*, 1993, **57**, 6, 51–60.
3. LAVROVSKII, E. K. and FORMAL'SKII, A. M., Stabilization of the equilibrium of a circular membrane. *Prikl. Mat. Mekh.*, 1997, **61**, 3, 457–465.
4. ZLOCHEVSKII, S. I. and KUBYSHKIN, Ye. P., Stabilization of an artificial satellite with flexible rods. *Kosmich. Issledovaniya*, 1991, **29**, 6, 828–839.
5. ROITENBERG, Ya. N., *Automatic Control*. Nauka, Moscow, 1992.
6. BITSADZE, A. V., *Fundamentals of the Theory of Analytic Functions of a Complex Variable*. Nauka, Moscow, 1969.
7. NURRE, G. S., RYAN, R. S., SCOFIELD, H. N. and SIMS, J. L., Dynamics and control of large space structures. *Guidance, Control, and Dynamics*, 1984, **7**, 5, 514–526.
8. BALLAS, M. J., Modal control of certain flexible dynamic systems. *SIAM J. Control and Optimization*, 1978, **16**, 3, 450–462.
9. MORRIS, K. A. and VIDYASAGAR, M., A comparison of different models for beam vibrations from the standpoint of control design. *Dynamics, Systems, Measurement, and Control*, 1990, **112**, 3, 356–394.

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